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One-dimensional directed sandpile models and the area under a Brownian curve

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Abstract
We derive the steady state properties of a general directed 'sandpile' model in one dimension. Using a central limit theorem for dependent random variables we find the precise conditions for the model to belong to the universality class of the totally asymmetric Oslo model, thereby identifying a large universality class of directed sandpiles. We map the avalanche size to the area under a Brownian curve with an absorbing boundary at the origin, motivating us to solve this Brownian curve problem. Thus, we are able to determine the moment generating function for the avalanche-size probability in this universality class, explicitly calculating amplitudes of the leading order terms.

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1. Introduction

Sandpile models have played an important role in developing our understanding of self-organized criticality [1–5]. One important notion is that of universality, the idea that quantities such as critical exponents and scaling functions are independent of microscopic details of the model. This has been studied in the context of individual models, but few have determined general conditions for models to belong to a particular universality class [2, 5–7]. In the following, we present details of the solution of a general directed one-dimensional sandpile model introduced in [8] which is a generalization of a model studied in [9, 10]. We use a central limit theorem for dependent random variables [11] to determine the precise microscopic conditions for scaling of the moments of the avalanche-size probability. We also argue that there is an n-dependent crossover length ξ_n, such that for systems with size $L \ll \xi_n$ branching process behaviour is observed.

The avalanche-size statistics are calculated by mapping the model to the problem of finding the area under a Brownian curve with an absorbing boundary at the origin, that is, if $x(t)$ is the trajectory of a Brownian curve such that if $x(t') = 0$ for some $t' > 0$ then

$x(t > t') \equiv 0$. In the large L limit, the avalanche-size statistics are identical to those for the area under the Brownian curve after a 'time' equal to L; $A = \int_0^L x(t)\,\mathrm{d}t$. This motivated us to calculate the moment generating function for this area, which is an interesting problem in its own right as there have been some recent interest in physical applications of the statistics of the area under Brownian curves [12–14]. This work also relates to that of [2], where it was found that the avalanche statistics for a two-dimensional sandpile were related to the area enclosed by two annihilating random walkers which start on the same lattice site.

2. Definition of the model

The model we study is on a one-dimensional lattice of length L where each lattice site, $i = 1, 2, \ldots, L$, may be in one of n states. The state of site i is denoted by z_i, which may take values $0, \ldots, n-1$, and this is interpreted as the number of particles on site i.

At the beginning of each time step a particle is added to site 1: $z_1 \to z_1 + 1$. This site may topple a number of times, each toppling redistributing one particle from site 1 to site 2: $z_1 \to z_1 - 1, z_2 \to z_2 + 1$. When site 2 receives a particle it may undergo topplings, redistributing particles to site 3, which in turn may topple, and so on until either a site does not topple, or site L topples where the redistributed particles leave the system and the time step ends. The avalanche size, s, is the total number of topplings which occur during a single time step. The toppling rules are therefore defined through choosing the probability that a site with z particles will topple so many times upon receiving a particle.

The only restrictions on the topplings are as follows: (i) z_i must remain in the range $[0, n-1]$. For instance, a site with $z_i = 2$ may not topple more than three times when receiving a particle. Moreover, a site with $z_i = n-1$ which receives a particle must topple at least once. (ii) When site i topples it redistributes exactly one particle to site $i+1$ only. (iii) The toppling rule is homogeneous and obeys a Markov property in that the probability that site i topples s_i times depends only on z_i. In fact, the requirement on the homogeneity can be relaxed and all the following is trivially extended to inhomogeneous toppling rules. Each site will then have a different stationary state, but provided that the remaining constraints are obeyed, the scaling of the avalanches will remain unaltered. (iv) There must be some probabilistic element to the toppling rules. To be precise, there must exist at least one value of z such that the number of topplings a site in this state undergoes is non-deterministic. This last restriction discounts purely deterministic toppling rules which lead to trivial dynamics.

Self-organized criticality (SOC) is associated with a stationary state where the avalanche-size probability, $P(s; L)$, which is the probability of observing an avalanche of size s in a system of size L, obeys simple finite-size scaling,

$$P(s; L) = as^{-\tau} G(s/bL^{\Delta}) \qquad \text{for} \quad s \gg 1, \quad L \gg 1, \tag{1}$$

where a and b are non-universal constants, τ and Δ are universal exponents and G is a universal scaling function. It can be shown that if $\lim_{L\to\infty} \int_{1/bL^{\Delta}}^{\infty} u^{k-\tau} G(u)\,\mathrm{d}u$ exists and is non-zero for $k = 1, 2, \ldots$, then

$$\langle s^k \rangle_L \equiv \sum_{s=1}^{\infty} s^k P(s; L) \propto L^{\Delta(k+1-\tau)}. \tag{2}$$

Hence, for $L \gg 1$, the scaling of the moments $\langle s^k \rangle_L$ with L is universal and we can determine the universality class of a model by calculating the exponents Δ and τ from the scaling of these moments.

To formulate the model, we use a Markov matrix representation, which is an extension to the work in [10]. The configuration of a system is $\{z_1, z_2, \ldots, z_L\}$ for which we shall use the shorthand notation $\{z_i\}$. In order to construct a Markov matrix representation, we first consider a representation for the configuration of a single site, that is, a system of size $L = 1$. Such a system can be in one of n stable configurations, $z = 0, 1, \ldots, n-1$, and therefore we require an n-dimensional vector space in order to define a probability measure over these configurations. We therefore construct n-dimensional left and right basis vectors, $\{\langle e_z|\}$ and $\{|e_z\rangle\}$, where $\langle e_z|$ and $|e_z\rangle$ are vectors with the $(z+1)$th element equal to 1 and all the rest equal to 0, such that $\langle e_z|e_{z'}\rangle = \delta_{z,z'}$. The configuration of a single site with z particles is then represented by the basis vector $|e_z\rangle_1 \equiv |e_z\rangle$, where the subscript 1 reminds us that it is the configuration of one single site. The n^L-dimensional vectors representing configurations $\{z_i\}$ for systems with L sites, $|e_{\{z_i\}}\rangle_L$, are constructed from these basis vectors

$$|e_{\{z_i\}}\rangle_L = \left|e_{z_1}\right\rangle_1 \otimes \left|e_{z_2}\right\rangle_1 \otimes \cdots \otimes \left|e_{z_L}\right\rangle_1, \tag{3}$$

where $\otimes$ is the usual tensor product.

A state vector $|P_t\rangle_L$ is the weighted sum of basis vectors,

$$|P_t\rangle_L = \sum_{\{z_i\}} w^t_{\{z_i\}} \left|e_{\{z_i\}}\right\rangle_L, \tag{4}$$

where the weights $w^t_{\{z_i\}}$ are the probabilities that the system is in the configuration $\{z_i\}$ at time t, that is $w^t_{\{z_i\}} = \left\langle e_{\{z_i\}}\middle|P_t\right\rangle$. Note that the normalization condition requires

$$\sum_{\{z_i\}} w^t_{\{z_i\}} = \sum_{\{z_i\}} \left\langle e_{\{z_i\}}\middle|P_t\right\rangle_L = 1. \tag{5}$$

The system is evolved by applying operators to the state vector $|P_t\rangle_L$. We define a toppling operator G_L which adds a particle to site 1 of a system of size L and carries out all the topplings:

$$|P_{t+1}\rangle_L = \mathsf{G}_L|P_t\rangle_L, \tag{6}$$

making it clear that $|P_t\rangle_L$ is a Markov chain. Using the $|e_z\rangle$ representation the toppling operator is an $n^L \times n^L$ matrix where $\langle e_{\{z_i'\}}|_L \mathsf{G}_L|e_{\{z_i\}}\rangle_L$ is the probability that adding a particle to a system in the configuration $|e_{\{z_i\}}\rangle_L$ results in the configuration $|e_{\{z_i'\}}\rangle_L$ after topplings. See appendix A for an explicit representation of G_L for a system with $n = 2$. The steady state, $|0\rangle_L$, is defined as the state which is invariant under application of the toppling operator G_L. It is therefore the right eigenvector of G_L with eigenvalue 1,

$$\mathsf{G}_L|0\rangle_L = |0\rangle_L. \tag{7}$$

The corresponding left eigenvector, $\langle 0|_L$, satisfies

$$\langle 0|_L \mathsf{G}_L = \langle 0|_L. \tag{8}$$

Due to conservation of probability, the sum of each column in G_L equals 1. This leads to

$$(1, 1, \ldots, 1)\mathsf{G}_L = (1, 1, \ldots, 1), \tag{9}$$

identifying the left eigenvector of the toppling operator $\langle 0|_L = (1, 1, \ldots, 1)$.

In order to calculate the moment generating function for the avalanche-size probability, it is convenient to define the operator $\mathsf{G}_L(x)$ such that

$$Q_{L,m}(x) = \langle 0|_L \left[\mathsf{G}_L(x)\right]^m |0\rangle_L \tag{10}$$

is the avalanche-size moment generating function over m time steps and

$$\langle s^k\rangle_L = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^k Q_{L,1}(x)\Bigg|_{x=1} \equiv Q^{(k)}_{L,1} \tag{11}$$

gives the kth moment of the avalanche-size probability in a system of size L. The toppling operator is then

$$\mathsf{G}_L \equiv \mathsf{G}_L(1). \tag{12}$$

To illustrate how to construct $\mathsf{G}_L(x)$, consider the action of adding a particle to site $i = 1$. First, we look at the number of particles on that site, z_1, and determine the number of times the site topples using the probabilities given by the toppling rule. Hence, we require $n + 1$ matrices of dimension n, denoted by S_k, which will act on the first site and remove k particles, multiplying the final state by the probability that this toppling took place. We then need to redistribute these particles to site $i = 2$, which we achieve by acting $(\mathsf{G}_{L-1})^k$ on the remaining $L - 1$ sites. This works because adding to site 2 of a system of size L is equivalent to adding to site 1 of a system of size $L - 1$ (note $\mathsf{G}_0(x) \equiv \mathbb{1}$). Finally, we multiply the remaining state by a factor x^k, which marks the state as having toppled k times, which gives moments of the avalanche size upon differentiation. This leads us to write the general toppling operator

$$\mathsf{G}_L(x) = \sum_{k=0}^{n} x^k [\mathbb{1} \otimes \mathsf{G}_{L-1}(x)]^k \mathsf{S}_k \otimes \mathbb{1}^{\otimes L-1}, \tag{13}$$

where $\mathbb{1}$ is an $n \times n$ identity matrix and $\mathsf{A}^{\otimes N} \equiv \mathsf{A} \otimes \mathsf{A} \otimes \cdots \otimes \mathsf{A}$, N times. The restrictions, (i)–(iv), on the model give $[\mathsf{S}_k]_{ij} \geqslant 0$ for $j = i + k - 1$ and equal to zero otherwise, that is,

$$\mathsf{S}_k = \sum_z |e_{z+k-1}\rangle_1 S_{z,z+k-1} \langle e_z|_1, \tag{14}$$

where the sum is over all z which satisfy both $0 \leqslant z \leqslant n - 1$ and $0 \leqslant z + k - 1 \leqslant n - 1$, and $S_{z,z+k-1}$ is the probability that a site with z particles topples k times on receiving a particle. Note that particle conservation requires $\sum_{j=0}^{n-1} S_{z,j} = 1$.

3. Stationary properties

In this section we find the steady state, $|0\rangle_L$, which is the eigenvector of G_L with eigenvalue 1. Consider the single site operator,

$$\mathsf{G}_1(x) = \sum_{k=0}^{n} x^k \mathsf{S}_k. \tag{15}$$

We begin by finding the eigenvectors and eigenvalues defined by

$$\langle \lambda_i(x)|\mathsf{G}_1(x) = \lambda_i \langle \lambda_i(x)| \tag{16a}$$

$$\mathsf{G}_1(x)|\lambda_i(x)\rangle = \lambda_i |\lambda_i(x)\rangle, \tag{16b}$$

where i takes values from 0 to $n-1$. From the properties of S_k and the normalization condition we find that

$$\langle \lambda_0(x)| = \left(\frac{1}{x^{n-1}}, \frac{1}{x^{n-2}}, \ldots, \frac{1}{x}, 1\right) \tag{17}$$

satisfies

$$\langle \lambda_0(x)| \sum_{k=0}^{n} x^k \mathsf{S}_k = x \langle \lambda_0(x)|, \tag{18}$$

and so is the left eigenvector of $\mathsf{G}_1(x)$ with eigenvalue $\lambda_0 = x$. The corresponding right eigenvector therefore satisfies

$$\mathsf{G}_1(x)|\lambda_0(x)\rangle = \sum_{k=0}^{n} x^k \mathsf{S}_k|\lambda_0(x)\rangle = x|\lambda_0(x)\rangle, \tag{19}$$

which determines the precise form of the eigenvector. As the eigenvectors must be normalized, $\langle\lambda_0(x)|\lambda_0(x)\rangle = 1$, we may write

$$|\lambda_0(x)\rangle = \begin{pmatrix} p_0 x^{n-1} \\ p_1 x^{n-2} \\ \vdots \\ p_{n-1} x^0 \end{pmatrix}, \tag{20}$$

where p_z is the probability that a site contains z particles in the stationary state and $\sum_{z=0}^{n-1} p_z = 1$. We cannot, however, determine p_i any more precisely without details of S_k and these will have to be calculated separately in each case.

If the matrix G_1 is a regular Markov matrix, that is, there exists an integer $N \geqslant 1$ such that $\left[\mathsf{G}_1^N\right]_{ij} > 0$ for all i, j, then we have found the unique stationary state of the single site operator,

$$|0\rangle = |\lambda_0(1)\rangle = \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{pmatrix}. \tag{21}$$

In the following we shall always assume that G_1 is regular. The discussion of the necessary and sufficient conditions for a regular G_1 is non-trivial and we shall not discuss it in detail. We shall simply note that this requirement, along with restrictions (i)–(iv), still leaves an abundance of choice for the toppling rules. For instance, it is easy to demonstrate that any tridiagonal matrix with positive definite elements is regular. Hence, any toppling rule which always allows a site to topple zero times, once or twice on receiving a particle (with the usual exceptions for $z = 0$ and $z = n - 1$) will automatically lead to an acceptable toppling rule.

Now that we have found the stationary state of the single site operator G_1, we shall proceed to determine the stationary properties of the full operator G_L by induction. We introduce the notation $|\lambda_i(x)\rangle_L$ as the ith eigenvector of $\mathsf{G}_L(x)$ such that

$$\mathsf{G}_L(x)|\lambda_i(x)\rangle_L = \lambda_{L,i}(x)|\lambda_i\rangle_L, \tag{22}$$

where $\lambda_{L,i}(x)$ is the ith eigenvalue of $\mathsf{G}_L(x)$. We now make the ansatz that these eigenvectors may be expressed as

$$|\lambda_{i+j}(x)\rangle_L = |\lambda_j(x\lambda_{L-1,i}(x))\rangle_1 \otimes |\lambda_i(x)\rangle_{L-1}. \tag{23}$$

Operating on the left-hand side with $\mathsf{G}_L(x)$ we find

$$\mathsf{G}_L(x)|\lambda_{i+j}(x)\rangle_L = \sum_{k=0}^{n} (x\lambda_{L-1,i})^k \mathsf{S}_k|\lambda_{i+j}(x)\rangle_L \tag{24}$$

$$= \mathsf{G}_1(x\lambda_{L-1,i})|\lambda_j(x\lambda_{L-1,i}(x))\rangle_1 \otimes |\lambda_i(x)\rangle_{L-1} \tag{25}$$

$$= x\lambda_j(x\lambda_{L-1,i})|\lambda_{i+j}(x)\rangle_L. \tag{26}$$

So, we find that $|\lambda_{i+j}(x)\rangle_L$ is indeed an eigenvector of $\mathsf{G}_L(x)$ with eigenvalue $\lambda_{L,i+j} = \lambda_j(x\lambda_{L-1,i})$. Hence, by induction, and recalling that we have assumed G_L is regular, the unique steady state is

$$|0\rangle_L = |0\rangle_1 \otimes |0\rangle_{L-1} = |0\rangle_1^{\otimes L}. \tag{27}$$

This is a product state, which means that it has no spatial correlations and indeed we find that avalanches in small systems are uncorrelated, leading to branching process behaviour. However, we will show that in larger systems temporal correlations develop which bring the avalanche behaviour away from the branching process to that characterized by the area under a Brownian curve.

4. Toppling probability distribution

In this section we define the toppling probability distribution and determine some of its properties which will be used in the following section. The toppling probability distribution, $P(s; L, m)$, is defined as the probability that a system of size L in the stationary state undergoes a total of s topplings on receiving m particles. In principle, it may be calculated from the moment generating function

$$P(s; L, m) = \frac{1}{s!}\frac{\mathrm{d}^s}{\mathrm{d}x^s}\langle 0|_L[\mathsf{G}_L(x)]^m|0\rangle_L\bigg|_{x=0} \tag{28}$$

although this is rarely a simple task in practice. Recall that we only consider toppling rules obeying the restrictions (i)–(iv) with a unique stationary state. First, we show that $P(s; L, m)$ has a mean value

$$Q^{(1)}_{L,m} = \sum_{s=0}^{\infty} sP(s; L, m) = mL, \tag{29}$$

which is what we would expect by considering conservation in the stationary state since every particle that enters the system must leave through the open boundary. We write down the equation for the first moment

$$Q^{(1)}_{L,m} = \frac{\mathrm{d}}{\mathrm{d}x}\langle 0|_L[\mathsf{G}_L(x)]^m|0\rangle_L\bigg|_{x=1} = m\sum_{k=0}^{n}\langle 0|_1 k\mathsf{S}_k|0\rangle_1\big(1 + Q^{(1)}_{L-1,1}\big). \tag{30}$$

Multiplying (19) on the left by $\langle\lambda_0(x)|$ and differentiating, we find

$$\langle\lambda_0(x)|\sum_{k=0}^{n} kx^{k-1}\mathsf{S}_k|\lambda_0(x)\rangle = 1 \tag{31}$$

and so $\langle 0|_1\sum_{k=0}^{n} k\mathsf{S}_k|0\rangle_1 = 1$ giving $Q^{(1)}_{L,m} = m\big(1 + Q^{(1)}_{L-1,1}\big)$, with $Q^{(1)}_{1,1} = 1$, which has the solution $Q^{(1)}_{L,m} = mL$.

Next, we show that the avalanche probability may be factorized. We define $P(s, t; m, 1, L)$ as the joint probability that a system of size $1 + L$, which has received m particles, undergoes s topplings in the first site and t in the remaining L sites:

$$P(s, t; m, 1, L) = \frac{1}{s!}\frac{\mathrm{d}^s}{\mathrm{d}x_1^s}\frac{1}{t!}\frac{\mathrm{d}^t}{\mathrm{d}x_L^t}\langle 0|_{L+1}[\mathsf{G}_{L+1}(x_1, x_L)]^m|0\rangle_{L+1}\bigg|_{x_1=0\,x_L=0}, \tag{32}$$

where

$$\mathsf{G}_{L+1}(x_1, x_L) \equiv \sum_{k=0}^{n} x_1^k\mathsf{S}_k \otimes [\mathsf{G}_L(x_L)]^k. \tag{33}$$

By expanding the bracket in (32) and carrying out the differentiation with respect to x_1, we find

$$P(s, t; m, 1, L) = \langle 0|_1 \sum_{\{k_i\}} \delta\left(\sum_i k_i - s\right) \prod_{i=0}^{m} \mathsf{S}_{k_i} |0\rangle_1 \frac{1}{t!} \frac{\mathrm{d}^t}{\mathrm{d}x_L^t}\bigg|_{x=0} \langle 0|_L \left(\mathsf{G}_L(x_L)\right)^s |0\rangle_L, \quad (34)$$

where $\delta(x)$ is the Kronecker delta, $\delta(0) = 1$ and $\delta(x) = 0$ for $x \neq 0$. Identifying the first scalar product as simply the probability that a site receiving m particles topples s times, $P(s; m, 1)$, we have

$$P(s, t; m, 1, L) = P(s; m, 1)P(t; s, L). \quad (35)$$

This is simply a statement about the fact that the directed nature means that sites $i = 2, \ldots, L$ do not have any influence on site $i = 1$. Hence, if we consider s_i, which is the number of times site i topples during a particular avalanche, this result tells us that in the stationary state the sequence $s_1, s_2, s_3, \ldots, s_L$ for a single avalanche forms a Markov chain. This will become important later on when we come to map the avalanche size $s = \sum_{i=1}^{L} s_i$ to the area under a random walker.

We now derive three important results for the single site toppling probability distribution, $P(s; 1, m)$, which we will use later to make the mapping of avalanches to the area under a Brownian curve more rigorous. First, we show that the range of s for which $P(s; 1, m)$ has support has an upper bound equal to $2n - 1$. Second, we show that $P(s; 1, m)$ has a stationary distribution for large m,

$$\lim_{m\to\infty} P(s; 1, m) = \sum_z p_z p_{(m-s)+z}, \quad (36)$$

where the sum is over all $0 \leqslant z \leqslant n - 1$ satisfying $0 \leqslant m - s + z \leqslant n - 1$. Note that this is a function of $m - s$ only. Finally, we consider the width of $P(s; 1, m)$ around its mean value,

$$\tilde{Q}^{(2)}_{1,m} \equiv \sum_{s=0}^{\infty} (s - m)^2 P(s; 1, m) \quad (37)$$

and show that $\tilde{Q}^{(2)}_{1,m} > 0$, for all m. These results lead to a further result that the width is finite and non-zero for all m, and approaches a constant for $m \to \infty$, which, again, will become important later when we map the problem to a random walker.

The first of these results follows immediately from bulk conservation of particles. Consider a site with z particles, which receives m particles. After s topplings have taken place it will have $z' = z+m-s$ particles. Since both z and z' must lie between 0 and $n-1$, $P(s; 1, m)$ may only have non-zero values for $m - n + 1 \leqslant s \leqslant m + n - 1$. Hence, $P(s; 1, m) = 0$ for $|s - m| > n - 1$ since such topplings are always impossible and $\tilde{Q}^{(2)}_{1,m} \leqslant (n - 1)^2 < \infty$ for $n < \infty$. Hence, $\tilde{Q}^{(2)}_{1,m}$ is finite because the sum in (37) has at most $2n - 1$ non-zero terms.

Next, we consider the probability distribution, $P_m(z'|z)$, which is the probability that a site having z particles which receives m particles is left with z' particles after $s = z' - z + m$ topplings. In the stationary state, this is related to $P(s; 1, m)$ by

$$P(s; 1, m) = \sum_{z=0}^{n-1} p_z P_m(m + z - s|z). \quad (38)$$

Each time a site receives one of the m particles, it will topple a number of times and the number of particles on the site, z, will change intermittently. The sequence of values z passes through forms a Markov chain. Hence, we may write down the Chapman–Kolmogorov equation for $P_m(z'|z)$,

$$P_{m+1}(z'|z) = \sum_{z''=0}^{n-1} P_1(z'|z'')P_m(z''|z). \tag{39}$$

The probabilities $P_1(z'|z'')$ are related to the single site operator, G_1 by $P_1(z'|z'') = \langle e_{z'}|_1\mathsf{G}_1|e_{z''}\rangle_1$ and so we introduce the matrix P_m, such that $P_m(z'|z'') = \langle e_{z'}|_1\mathsf{P}_m|e_{z''}\rangle_1$ and (39) becomes

$$\langle e_{z'}|_1\mathsf{P}_{m+1}|e_z\rangle_1 = \sum_{z''=0}^{n-1}\langle e_{z'}|_1\mathsf{G}_1|e_{z''}\rangle_1\langle e_{z''}|_1\mathsf{P}_m|e_z\rangle_1 = \langle e_{z'}|_1\mathsf{G}_1\mathsf{P}_m|e_z\rangle_1. \tag{40}$$

Thus $\mathsf{P}_{m+1} = \mathsf{G}_1\mathsf{P}_m$ and, since G_1 is a regular Markov matrix, there will be a unique stationary distribution P_∞ satisfying

$$\mathsf{P}_\infty = \mathsf{G}_1\mathsf{P}_\infty. \tag{41}$$

The only non-trivial solution to (41) is

$$\mathsf{P}_\infty = (|0\rangle_1, |0\rangle_1, \ldots, |0\rangle_1), \tag{42}$$

where we have noted that the stationary state, $|0\rangle_1$, is unique and conservation of probability requires $\sum_{z'=0}^{n-1}\langle e_{z'}|_1\mathsf{P}_\infty|e_z\rangle_1 = 1$. This leads us to

$$\lim_{m\to\infty} P_m(z'|z) = p_{z'} \tag{43a}$$

$$\lim_{m\to\infty} P(s; 1, m) = \sum_z p_z p_{(m-s)+z}, \tag{43b}$$

where we have noted that $\langle e_{z'}|0\rangle_1 = p_{z'}$ and the sum is over $0 \leqslant z \leqslant n-1$ satisfying $0 \leqslant m - s + z \leqslant n-1$. Finally, (43*b*) leads to

$$\lim_{m\to\infty} \tilde{Q}^{(2)}_{1,m} = \sum_{s=-n+1}^{n-1} s^2 \sum_z p_z p_{z-s}, \tag{44}$$

where the sum is over all $0 \leqslant z \leqslant n-1$ such that $0 \leqslant z + s \leqslant n-1$. This limit exists, is non-zero and is finite for $n < \infty$.

Finally, we show that $\tilde{Q}^{(2)}_{1,m} > 0$ for all m. Consider again the probability distribution $\mathsf{P}_m = [\mathsf{G}_1]^m$ and the equation for the width,

$$\tilde{Q}^{(2)}_{1,m} = \sum_{s=0}^{\infty}(s-m)^2 \sum_{z=0}^{n-1} p_z P_m(m+z-s|z). \tag{45}$$

For $\tilde{Q}^{(2)}_{1,m^*} = 0$, for some m^*, we require

$$p_z\langle e_{z'}|_1\mathsf{P}_{m^*}|e_z\rangle_1 = p_z\delta_{z,z'}, \tag{46}$$

which follows from normalization of P_m and the fact that (45) has no negative terms. This implies that, for all $0 \leqslant z \leqslant n-1$ for which $p_z > 0$,

$$[\mathsf{G}_1]^{Nm^*}|e_z\rangle_1 = |e_z\rangle_1, \tag{47}$$

where $N > 0$ is an integer. However, since G_1 is regular, there exists an integer N^* such that there is only one vector, $|0\rangle_1$, satisfying

$$[\mathsf{G}_1]^N|0\rangle_1 = |0\rangle_1 \tag{48}$$

for any $N > 0$. If there is more than one value of z for which $p_z > 0$ this contradicts (47), and so $\tilde{Q}^{(2)}_{1,m}$ is never zero. If, however, we have a single value, z^*, such that $p_z = \delta_{z,z^*}$, then $|0\rangle_1 = |e_{z^*}\rangle$ and (47) does not lead to a contradiction. However, in this case the dynamics are trivial as the steady state has all sites with exactly z^* particles and any particle added to the system will pass through immediately with exactly L topplings.

5. Mapping to area under random walker

We now come to the main result we need in order to determine the avalanche statistics for the directed sandpile, which is that it may be mapped exactly onto a random walker on $[0, \infty)$ with an absorbing boundary at the origin. After adding a particle at the beginning of a time step, site $i = 1$ will topple $s_1 \geqslant 0$ times with probability $P(s_1; 1, 1)$. These s_1 particles are redistributed to site $i = 2$, which will topple $s_2 \geqslant 0$ times with probability $P(s_2; 1, s_1)$. The probability of site 2 toppling s_2 times, independent of s_1, which we denote $\phi_2(s_2)$, is

$$\phi_2(s_2) = \sum_{s_1=1}^{\infty} P(s_2; 1, s_1), \tag{49}$$

which follows from (35). Defining $\phi_i(x)$ as the probability that site i topples x times independent of previous topplings, we have

$$\phi_{i+1}(x) = \sum_{y=1}^{\infty} \phi_i(y) P(x; 1, y) \qquad \text{for} \quad i = 1, \ldots, L-1. \tag{50}$$

This is a random walker on the interval $[0, \infty)$ with the probability of hopping from y to x equal to $P(x; 1, y)$. There is an absorbing boundary at $x = 0$ since any non-toppling site stops the avalanche. If we denote the trajectory by $x(i)$, $i = 0, \ldots, L$, then the avalanche size is

$$s = \sum_{i=1}^{L} x(i) \tag{51}$$

with $x(0) = 1$, which is the area under the trajectory $x(i)$.

Note that the random walker described by (50) has jumps which are correlated since the probability of hopping from y to x depends explicitly on y and x, and not simply the difference $x - y$. This means we must be careful if we wish to use the results for the uncorrelated random walker, or its continuum limit. However, in [11], the author remarks that for martingales with a fixed maximum jump size exhibiting stationarity and ergodicity, there is a quantity, $s_i^2 = \mathbb{E} \sum_{n=1}^{i} \sigma_n^2$, such that

$$\lim_{i\to\infty} P[x(\mathrm{i})/s_i \leqslant x] = (2\pi)^{-1} \int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}y^2}\, \mathrm{d}y, \tag{52}$$

where σ_n^2 is the variance of the nth step in the process.

To apply this result, we extend the random walker described by $P(x; 1, y)$ to the full space, $(-\infty, \infty)$ for we may add in the effect of the boundaries later by use of mirror charges [15]. As we have assumed the existence of a unique stationary state and have proven that $Q_{1,m}^{(1)} = m$ and $0 < \tilde{Q}_{1,m}^{(2)} \leqslant (n-1)^2$, all that is left to prove is ergodicity. This is equivalent to showing that the set of recurrent states of the random walker are irreducible, that is, the probability of reaching any recurrent state i from any other recurrent state j is non-zero. Two states, i and j, which have this property are said to intercommunicate, denoted by $i \leftrightarrow j$. We consider the fact that G_1 is assumed to be regular, in which case there exists an N such that $\langle e_z|[\mathsf{G}_1]^m|e_{z'}\rangle > 0$ for all $z, z' \in [0, n-1]$ and $m \geqslant N$. Hence, all states $i, j \geqslant N$ intercommunicate since $P(m \pm 1; 1, m) > 0$ for all $m \geqslant N$. We also note that $0 \leftrightarrow N$ and $1 \leftrightarrow N$ which follow respectively because the avalanche should always be able to finish in an infinite system, and arbitrarily large avalanches can be initiated from a single added particle. When we consider states $k < N$, we note that there can only be a finite number of these which do not intercommunicate with state 1. Since there is a unique stationary state which includes all states $i \geqslant N$, these non-intercommunicating states must be transient and ergodicity follows.

Hence, we have now proven that for a toppling rule obeying the restrictions (i)–(iv) with a unique stationary state (i.e. G_1 is regular), the distribution of the random walker on $(-\infty, \infty)$ will approach the normal distribution given by (52). This means that, for long times, such a random walker with dependent jump sizes will have the statistics of ordinary diffusion with diffusion constant $2D = s_n$. Hence, by adding mirror charges to remove paths that cross $x = 0$, we are able to calculate the large L statistics of avalanches directly from the area under the Brownian curve, which is our justification for calculating moments in the continuum limit in the following section. Of course, we could have simply gone ahead and carried out the calculations in the continuum without the above analysis and demonstrated that they correctly modelled the numerics. However, had we done so we would not have had a precise idea of how trustworthy these calculations were and where we expect them to break down.

6. Moments of the area under the Brownian curve

Having proven the correspondence between avalanches and a random walk of independent identically distributed step sizes, we proceed to calculate the moment generating function for the area under the Brownian curve. The authors are aware of only one study which investigates the finite-size effects due to stopping the curve after some time, which corresponds to the finite size of the sandpile [10] and since our analysis goes further than that in [10], we present it here in some detail. The following calculation will be carried out using notation and language suitable for the random walker description of the problem. Hence, the Brownian curve will be described by a trajectory $x(t)$ where x is interpreted as 'space' and t is 'time' with the diffusion constant D having units Length^2 Time^{-1}. We do this because the path integral approach we are about to employ is more intuitive in this language[1]. The connection to the sandpile is made by noting that the number of topplings of site i is equal to $x(t = i)$ and the system size, L, is equal to the time at which we stop the curve $x(t)$. We begin with the generating function

$$\langle \mathrm{e}^{-\lambda A}\rangle = \int_0^\infty \mathrm{e}^{-\lambda A} P(A; x, L)\,\mathrm{d}A, \tag{53}$$

where $P(A; x, L)$ is the probability that a random walker starting at x has the area under its trajectory equal to A after time L. If we denote the trajectory of the walker $x(t)$, then the curves contributing to $P(A; x, L)$ are all those which satisfy

$$\int_0^L x(t)\,\mathrm{d}t = A. \tag{54}$$

Note that we have an absorbing boundary at $x = 0$, such that if $x(t') = 0$ for any t then $x(t > t') \equiv 0$. Hence, there are two contributions to $P(A; x, L)$: that due to trajectories which do not cross the absorbing boundary, $x(L) > 0$ and those which cross the absorbing boundary at some time $t \leqslant L$, see figure 1.

We shall treat these separately, writing

$$\langle \mathrm{e}^{-\lambda A}\rangle = \int_0^\infty \mathrm{d}A \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-\lambda A}\Psi(A, y, L; x) + \int_0^\infty \mathrm{d}A \int_0^L \mathrm{d}t\, \mathrm{e}^{-\lambda A}\Phi(A, t; x) \equiv I_1 + I_2, \tag{55}$$

[1] We should stress that in this picture we consider the Brownian curve $x(t)$ as existing on the entire interval $[0, \infty)$ and we measure the area up to the point L, $A = \int_0^L x(t)\,\mathrm{d}t$. Hence, what is a boundary in the sandpile picture (the open boundary at site $i = L$) is not considered a boundary in the Brownian curve picture.

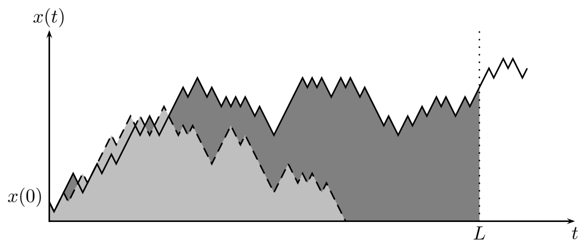

Figure 1. The area under the Brownian curve with an absorbing boundary. The statistics consist of contributions due to curves $x(t)$ which are non-zero at $t = L$ (solid line), as well as those which cross the boundary at some time $t < L$ (dashed line).

where $\Psi(A, y, L; x)$ is the probability that a trajectory beginning at $x(0) = x$ passes through $x(L) = y$ with area A, and $\Phi(A, t; x)$ is the probability that a trajectory beginning at $x(0) = x$ first touches the absorbing boundary at time t, with area A.

Using standard path integral methods, we may write down

$$\Psi(A, y, T; x) = \lim_{g\to\infty} \int_{\substack{x(0)=x\\ x(L)=y}} \mathcal{D}x(t)\delta\left(\int_0^L x(t)\,\mathrm{d}t - A\right) \exp\left(-\int_0^T [D\dot{x}^2 + g\delta(x)]\,\mathrm{d}t\right), \tag{56}$$

where $\dot{x} = \mathrm{d}x(t)/\mathrm{d}t$. Taking the integral over A we find that the first term on the right-hand side of (55) is

$$I_1 = \lim_{g\to\infty} \int_0^\infty \mathrm{d}y \int_{\substack{x(0)=x\\ x(L)=y}} \mathcal{D}x(t) \exp\left(-\int_0^L [D\dot{x}^2 + \lambda x + g\delta(x)]\,\mathrm{d}t\right). \tag{57}$$

Following the lines of [12], we note that this is simply the path integral for a Brownian particle with a linear potential for $x \in (0, \infty)$ and an infinite potential at $x = 0$. Hence, we write this term as

$$I_1 = \lim_{g\to\infty} \int_0^\infty \langle y|\,\mathrm{e}^{-\hat{H}L}|x\rangle\,\mathrm{d}y, \tag{58}$$

where $\hat{H} = -D\frac{\partial^2}{\partial x^2} + \lambda x + g\delta(x)$. The resulting equation of motion, $\frac{\partial}{\partial t}|\phi\rangle = -\hat{H}|\phi\rangle$, is easily solved using Airy functions which can be used to form an orthonormal basis on $[0, \infty)$ [16],

$$I_1 = \sum_{j=1}^\infty \frac{\mathrm{Ai}\big(\big(\frac{\lambda}{D}\big)^{1/3}x + x_j\big)\int_{x_j}^\infty \mathrm{Ai}(z)\,\mathrm{d}z}{\mathrm{Ai}'(x_j)^2}\,\mathrm{e}^{x_j\lambda^{2/3}D^{1/3}L}, \tag{59}$$

where x_j are the zeros of the Airy function, $x_1 = -2.338\ldots$, $x_2 = -4.087\ldots$ etc.

In a similar way, for the second term on the right-hand side of (55), we have

$$\Phi(A, t; x) = D\,\frac{\partial}{\partial y}\Psi(A, y; x, t)\bigg|_{y=0}, \tag{60}$$

since this is the current of diffusing particles with area under the curve equal to A, leaving the system at time t. Hence

$$I_2 = \int_0^L \frac{\partial}{\partial y}\langle y|\,\mathrm{e}^{-\hat{H}t}|x\rangle\bigg|_{y=0}\mathrm{d}t = \int_0^L \lambda^{2/3}D^{1/3}\sum_{j=1}^\infty \frac{\mathrm{Ai}\big(\big(\frac{\lambda}{D}\big)^{1/3}x + x_j\big)}{\mathrm{Ai}'(x_j)}\,\mathrm{e}^{x_j\lambda^{2/3}D^{1/3}t}\,\mathrm{d}t. \tag{61}$$

In order to proceed, we use the fact that the leading order L dependence for each moment comes from terms linear in x. In [10] it was shown that if the moment generating function is written as

$$\langle A^n\rangle = (-1)^n \frac{\partial^n}{\partial\lambda^n}\langle \mathrm{e}^{-\lambda A}\rangle\bigg|_{\lambda=0} \equiv n!\psi_n(x, L), \tag{62}$$

then $\phi_n(x, L)$ may be determined recursively

$$\psi_n(x, L) = \int_0^\infty \mathrm{d}x' \int_0^L \mathrm{d}t\, x'\psi_{n-1}(x', t)G(x, L-t; x'), \tag{63}$$

where $G(x, t; x')$ is the propagator for the diffusion equation with appropriate boundaries,

$$G(x, t; x') = \frac{\mathrm{e}^{-\frac{(x-x')^2}{4Dt}} - \mathrm{e}^{-\frac{(x+x')^2}{4Dt}}}{\sqrt{4\pi Dt}}. \tag{64}$$

If we define the 'current'

$$j_n(L) = \frac{\partial}{\partial x}\psi_n(x, L)\bigg|_{x=0}, \tag{65}$$

then if $j_n(L)$ is non-zero, $\psi_n(x, L)$ is proportional to x to lowest order. In this case

$$j_n(L) = \int_0^\infty \mathrm{d}x' \int_0^L \mathrm{d}t \frac{x'^2}{\sqrt{4\pi}} \frac{\psi_{n-1}(x', t)}{[D(L-t)]^{3/2}}\, \mathrm{e}^{-\frac{x'^2}{4D(L-t)}} \tag{66}$$

and so $j_n(L) > 0$ for $n > 0$ since the integrand is always positive definite. Hence, all moments are proportional to x to lowest order. The fact that the terms linear in x will also be the highest order in L follows from dimensional analysis. If we write down the expansion of a moment in powers of x, then each term must have the same dimension. By considering the dimensions of the available quantities, such an expansion must take the form

$$\langle A^n\rangle = xC_nD^{(n-1)/2}L^{(3n-1)/2} + \sum_{k=2}^{\infty} x^kC_{n,k}D^{(n-1-2k)/2}L^{(3n-1-2k)/2}, \tag{67}$$

where $C_{n,k}$ are simply more coefficients with no x, D or L dependence. Hence, the term of lowest order in x will have the highest order L dependence.

Taylor expanding I_1 and I_2 to first order about $x = 0$,

$$I_1 \approx x\left(\frac{\lambda}{D}\right)^{1/3} \sum_{j=1}^{\infty} \frac{\int_{x_j}^\infty \mathrm{d}z\,\mathrm{Ai}(z)}{\mathrm{Ai}'(x_j)}\, \mathrm{e}^{x_j\lambda^{2/3}D^{1/3}L} \equiv J_1 \tag{68a}$$

$$I_2 \approx x\lambda \int_0^L \sum_{j=0}^{\infty} \mathrm{e}^{x_j\lambda^{2/3}D^{1/3}t}\,\mathrm{d}t \equiv J_2. \tag{68b}$$

Note, however, that this approximation is not valid for the zeroth moment, $\langle A^0\rangle = 1$, since it is not proportional to x. J_1 and J_2 are now in similar forms to equations appearing in [12]. They calculate the quantity

$$\tilde{P}(\lambda, L) = \sqrt{\pi}2^{-1/6}(\lambda L^{3/2})^{1/3} \sum_{j=1}^{\infty} \frac{\int_{x_j}^\infty \mathrm{Ai}(z)\,\mathrm{d}z}{\mathrm{Ai}'(x_j)}\, \mathrm{e}^{x_j\lambda^{2/3}2^{-1/3}L} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} a_n L^{3n/2}, \tag{69}$$

where a_n have been calculated in [17]. We simply quote the first few values,

$$a_0 = 1, \qquad a_1 = \frac{3}{4}\sqrt{\frac{\pi}{2}}, \qquad a_2 = \frac{59}{60}, \qquad a_3 = \frac{465}{512}\sqrt{\frac{\pi}{2}}, \qquad a_4 = \frac{5345}{3696}. \tag{70}$$

Apart from a few multiplicative prefactors, (69) differs from (68*a*) only by the fact that the former uses $D = 1/2$. We therefore have to reinsert the diffusion constant, D, which they assumed equal to $1/2$, but this is easily done by considering the dimensions of the results. We note that J_1 is a dimensionless function, and so

$$J_1 = \left(\frac{\frac{1}{2}}{D}\right)^{\gamma} x \left(\frac{\lambda}{D}\right)^{1/3} \frac{2^{1/6}}{\sqrt{\pi}} (\lambda L^{3/2})^{-1/3} \tilde{P}(\lambda, L), \tag{71}$$

where γ is chosen such that J_1 is dimensionless. It is then easy to show that $\gamma = 1/6 - n/2$ and

$$J_1 = x \sum_0^{\infty} \frac{(-\lambda)^n}{n!} c_n D^{(n-1)/2} L^{(3n-1)/2}, \tag{72}$$

where

$$c_n = \frac{2^{n/2} a_n}{\sqrt{\pi}}. \tag{73}$$

We may carry out an identical procedure for J_2. The equivalent quantity in [12] is

$$\tilde{P}(\lambda, L) = \sqrt{2\pi}(\lambda L^{3/2}) \sum_{j=0}^{\infty} \mathrm{e}^{x_j \lambda^{2/3} 2^{-1/3} L} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} b_n L^{3n/2}, \tag{74}$$

where, again, b_n have been calculated in [17], the first few values being

$$b_0 = 1, \qquad b_1 = \frac{1}{2}\sqrt{\frac{\pi}{2}}, \qquad b_2 = \frac{5}{12}, \qquad b_3 = \frac{15}{64}\sqrt{\frac{\pi}{2}}, \qquad b_4 = \frac{221}{1008}. \tag{75}$$

Following the same steps as above we find

$$J_2 = x \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} 2^{n/2} \frac{b_n}{2\sqrt{\pi}} D^{(n-1)/2} \int_0^L t^{3(n-1)/2}\, \mathrm{d}t = x \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} d_n D^{(n-1)/2} L^{(3n-1)/2}, \tag{76}$$

where

$$d_n = 2^{n/2} \frac{b_n}{(3n-1)\sqrt{\pi}}. \tag{77}$$

Hence we have

$$\langle \mathrm{e}^{-\lambda A} \rangle = 1 + x \sum_{n=1}^{\infty} C_n \frac{(-\lambda)^n}{n!} D^{(n-1)/2} L^{(3n-1)/2} + \mathrm{O}(x^2), \tag{78}$$

where $C_n = c_n + d_n$ and the first few values are

$$C_1 = 1, \qquad C_2 = \frac{32}{15\sqrt{\pi}}, \qquad C_3 = \frac{15}{8}, \qquad C_4 = \frac{4064}{693\sqrt{\pi}}. \tag{79}$$

The first two values are in perfect agreement with those derived in [10], and the authors are unaware of any previous calculations of C_n for $n > 2$. Thus we may immediately identify the exponents $\tau = 4/3$ and $\Delta = 3/2$, and the amplitudes allow us to compute universal amplitude ratios, which we will use later to compare the numerics against theory.

6.1. Crossover from branching process

The convergence of $\phi_i(x)$ to the normal distribution occurs only as $i \to \infty$, and hence the results above are only valid for $L \to \infty$. In using the Brownian curve instead of the exact curve described by $P(s; 1, m)$ we have taken a hydrodynamic limit and therefore thrown away any information about the statistics of the process for small L. It is natural, therefore, to ask how we expect the results to differ in this regime. We propose the existence of an n-dependent crossover length, ξ_n, such that the above scaling analysis is valid for $L \gg \xi_n$. We argue that for smaller systems, $1 \ll L \ll \xi_n$, we expect to see scaling corresponding to the branching process.

Consider adding a particle to the first site. If the probability that the site has z particles, p_z, has support for all $z \in [0, n-1]$, then for $n \gg 1$ it is likely that $0 \ll z \ll n-1$. In this regime we may assume that the number of times the site topples due to this added particle, which we denote by s_1, is largely independent of z. Site 2 therefore receives s_1 particles, each of which may cause it to topple s_2^j times, $i = 1 \ldots s_1$, with the total number of topplings of site 2, $s_2 = \sum_j s_2^j$. While z remains far from 0 and $n-1$, s_2^j will be largely uncorrelated and by continuing this argument to more sites, we see that while s_i remain small each site will topple nearly independently. However, as we continue through the system to higher i the s_i will begin to see large fluctuations and the avalanches will become correlated, assuming the scaling of the previous section. Hence, we argue that for systems with $1 \ll L \ll \xi_n$, the avalanches will resemble those of the uncorrelated branching process with exponents $\tau = 3/2$ and $\Delta = 2$ [18]. For larger systems, $L \gg \xi_n$, temporal correlations emerge in the avalanches and $\tau = 4/3$, $\Delta = 3/2$.

The fact that the above argument relies on realizations where p_z has support for a large range of z indicates that the crossover length ξ_n depends on the details of the toppling rules and as such cannot be thought to have any 'universal' qualities. Indeed, we have not specified how the toppling rules in a realization should be altered as n is increased, and so it is impossible to say anything *a priori* about the behaviour of ξ_n.

7. Numerics

We now support our claims with numerics by demonstrating that the correct scaling (with crossovers—see previous section) occurs for a particular realization of this directed sandpile model. In order to study the scaling we choose a realization such that it is clear how to generalize to higher n. The only remaining difficulty is to find the correct variance to put into the equations when we come to compare with numerics. In all that follows, we use $2D = \tilde{Q}^{(2)}_{1,\infty}$ as we find that it fits the data very well.

We compare the scaling predicted above with numerics from a realization with the following toppling rules: a site i, $1 < z_i < n-1$, which receives a particle will topple one, two or three times with probability $1/8$ or will not topple with probability $5/8$. A site with $z_i = 0$ will topple once with probability $3/8$, a site with $z_i = 1$ will topple once with probability $2/8$ and twice with probability $1/8$ and a site with $z_i = n-1$ will topple once with probability $6/8$, and two or three times each with probability $1/8$. A site with $z_i = n-1$ has to topple at least once in accordance with restriction (i).

We expect $\langle s^2 \rangle$ to scale with the system size

$$\langle s^2 \rangle_L \sim \begin{cases} L^3 & 1 \ll L \ll \xi_n, \\ L^{5/2} & L \gg \xi_n \end{cases} \tag{80}$$

where ξ_n is a correlation length with some (as yet unknown) n dependence. These results have been confirmed and are shown in figure 2.

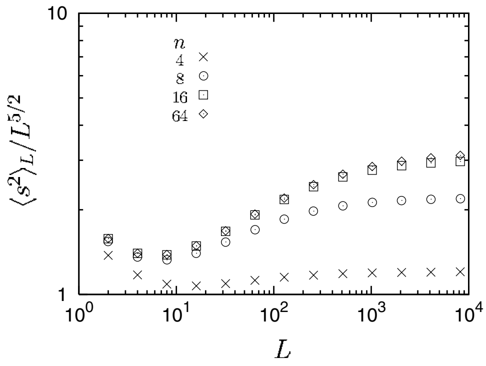

Figure 2. Numerical results for $n = 4, 8, 16, 64$. The errors for both graphs were calculated using Efron's Jackknife [19]. (*a*) The rescaled second moment $\langle s^2\rangle_L$ versus system size. For large systems this is a constant for all values of n. (*b*) The moment ratio $g_3(L)$. For large L this approaches the constant value $g_3 \approx 1.29$ for all values of n.

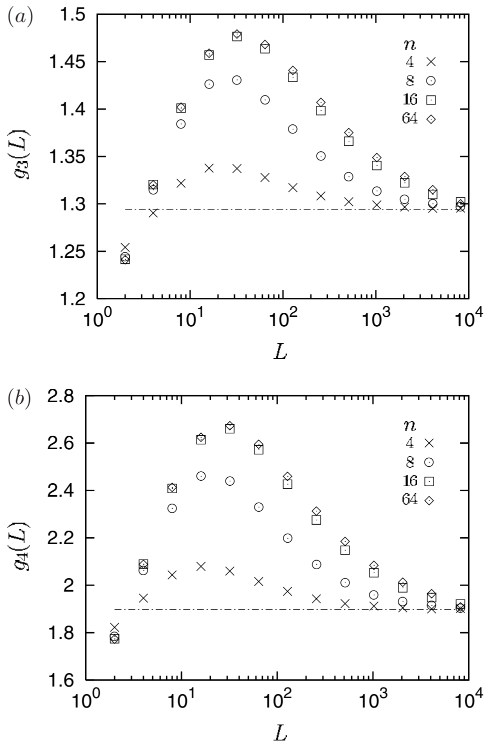

Figure 3. Numerical results for $n = 4, 8, 16, 64$. The errors for both graphs were calculated using Efron's Jackknife [19] and are approximately the same size as the symbols. The moment ratios $g_3(L)$ (*a*) and $g_4(L)$ (*b*). For large L these approach the constant values $g_3 \approx 1.29$ and $g_4 \approx 1.9$ respectively for all values of n. The dashed lines indicate the exact values $g_3 = 1.2942\ldots$ and $g_4 = 1.8975\ldots$, in excellent agreement with the numerics.

We also analyse the moment ratios defined by

$$g_k(L) = \frac{\langle s^k \rangle_L \langle s \rangle_L^{k-2}}{\langle s^2 \rangle_L^{k-1}}. \tag{81}$$

It is a straightforward calculation to show that, for an avalanche probability given by (1), $g_k(L)$ approach universal values for $L \to \infty$. These values are simply ratios of the amplitudes C_n calculated in section 6,

$$g_k \equiv \lim_{L\to\infty} g_k(L) = \frac{C_k}{C_2^{k-1}}. \tag{82}$$

This agrees with the numerics, as illustrated in figure 3 for $g_3(L)$ which appears to converge to a universal value of $g_3(\infty) \approx 1.29$, in excellent agreement with the theoretical prediction $g_3 = 15^3\pi/2^{13} = 1.294\ldots$ as well as numerics for a different realization published elsewhere [8]. This supports our claim that the $L \to \infty$ limits of the $g_k(L)$ are indeed universal. Note also that ξ_n has a notably different n dependence in this model than in the one presented in [8]. In this case, ξ_n saturates to a constant value for large n, meaning that for large n the crossover occurs at the same value of L. This is because the support of p_z is finite for $n \to \infty$.

8. Conclusion

We have found the stationary state avalanche-size distribution for a general n-state directed sandpile model. The avalanches can be mapped onto a random walk of dependent random variables and, using an applicable central limit theorem, we have shown that under a broad set of conditions the moments scale with $\tau = 4/3$ and $\Delta = 3/2$. We also note that this value of τ agrees precisely with that obtained in [13], which calculates the probability distribution in the infinite system size limit. We have also calculated the moment generating function for the area under a random walker with an absorbing boundary, and found a relation for the moment amplitudes in terms of those already known for other Brownian processes.

Acknowledgments

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Appendix A. Explicit example: $n = 2$

In this section we calculate the steady state properties of an $n = 2$ model, which is a generalization of the model studied in [10], and compare predictions to numerical simulation. For $n = 2$, the most general model we can write down, which obeys the rules (i)–(iv), is

$$\mathsf{S}_0 = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}, \tag{A.1}$$

$$\mathsf{S}_1 = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 1-\beta \end{pmatrix}, \tag{A.2}$$

$$\mathsf{S}_2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \tag{A.3}$$

where $0 \leqslant \alpha \leqslant 1$ is the probability that a site with $z = 0$ does not topple on receiving a particle and $0 \leqslant \beta \leqslant 1$ is the probability that a site with $z = 1$ topples twice on receiving a

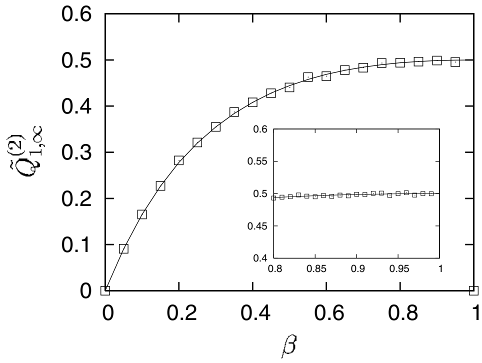


Figure A1. Numerical results for $\tilde{Q}^{(2)}_{1,\infty}$ for $\alpha = 1$ as a function of β with data (squares) compared against the values predicted by (A.10) (solid line). The data were obtained by measuring $\tilde{Q}^{(2)}_{1,m}$ for large m and estimating the asymptotic value. Comparison is made across the whole range of β and the inset shows data in the vicinity $\beta \to 1$. Note that the agreement is excellent right up to $\beta = 1$. Typical error bars for the numerical data are the size of the squares.

particle. Hence, the single site toppling matrix is

$$\mathsf{G}_1(x) = \begin{pmatrix} x - \alpha x & \beta x^2 \\ \alpha & x - \beta x \end{pmatrix}. \tag{A.4}$$

We now proceed to calculate the steady state properties of this model, following the prescription given in 3. $\mathsf{G}_1(x)$ has eigenvalues $\lambda = x$, $\mu = x(1 - \alpha - \beta)$ and eigenvectors

$$|e_\lambda(x)\rangle = \frac{1}{\alpha + \beta}\begin{pmatrix} \beta x \\ \alpha \end{pmatrix}, \tag{A.5}$$

$$\langle e_\lambda(x)| = \left(\frac{1}{x}, 1\right), \tag{A.6}$$

$$|e_\mu(x)\rangle = \begin{pmatrix} -x \\ 1 \end{pmatrix}, \tag{A.7}$$

$$\langle e_\mu(x)| = \frac{1}{\alpha + \beta}\left(\frac{-\alpha}{x}, \beta\right). \tag{A.8}$$

Hence, the eigenvector for the stationary state is

$$|0\rangle_L = |e_\lambda(0)\rangle^{\otimes L} = \left(\frac{1}{\alpha + \beta}\begin{pmatrix} \beta \\ \alpha \end{pmatrix}\right)^{\otimes L} \tag{A.9}$$

valid for $|\mu| \neq 1$.

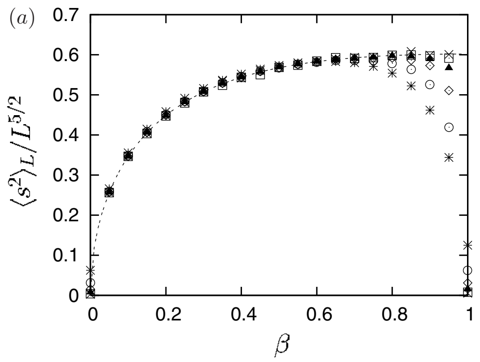
From these results it follows immediately:

$$\tilde{Q}^{(2)}_{1,\infty} = p_0 p_1 + p_1 p_0 = 2\frac{\alpha\beta}{\alpha + \beta} \tag{A.10}$$

and hence, from (78) and using $2D = \tilde{Q}^{(2)}_{1,\infty}$,

$$\langle s^2 \rangle_L \sim \frac{32}{15(\alpha + \beta)}\sqrt{\frac{\alpha\beta}{\pi}} L^{5/2}, \tag{A.11}$$

in perfect agreement with numerics, see figures A1 and A2. However, it should be noted that for α and β both approaching 1 the random walker it describes will spend more and more time



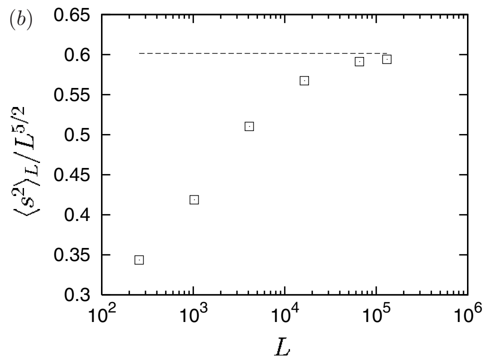


Figure A2. Numerical results for the rescaled second moment $\langle s^2\rangle_L/L^{5/2}$ for $\alpha = 1$ (*a*) as a function of β for $L = 256$ ($*$), 1024 ($\circ$), 4096 ($\diamond$), 16 385 ($\blacktriangle$), 65 536 ($\square$) and 131 072 ($\times$). (*b*) Rescaled second moment $\langle s^2\rangle_L/L^{5/2}$ for $\beta = 0.95$ versus inverse system size. The dashed line is the theoretical value. The measurements appear to converge towards the theoretical line large L, supporting our claim that the deviation is a finite size effect.

on either only odd or only even sites. Hence, it will take longer times (larger system sizes) for the statistics to reach the asymptotic values and so we expect very strong corrections to scaling for $\alpha, \beta \to 1$. When $\alpha = \beta = 1$, we no longer have a unique stationary state and so scaling is not observed.

Appendix B. Calculating the amplitudes C_n

The amplitudes a_n and b_n appearing in 6 can be calculated using the methods outlined in [12, 17]. For a_n we define

$$a_n = (\sqrt{2})^{-n}\frac{\Gamma(1/2)n!}{\Gamma\left(\frac{3n+1}{2}\right)}R_n, \tag{B.1}$$

where R_n are constructed through the following recursion relations:

$$R_n = \beta_n - \sum_{j=1}^{n}\gamma_j R_{n-j} \tag{B.2}$$

$$\beta_n \equiv \gamma_n + \frac{3}{4}(2n-1)\beta_{n-1} \tag{B.3}$$

Table B1. Tabulated values of C_n and g_n.

n	C_n	g_n
1	1	1
2	$\frac{32}{15\sqrt{\pi}}$	1
3	$\frac{15}{8}$	$\frac{3375}{8192}\pi$
4	$\frac{4096}{693\sqrt{\pi}}$	$\frac{47\,625}{78\,848}\pi$
5	$\frac{2875}{448}$	$\frac{145\,546\,875}{469\,762\,048}\pi^2$
6	$\frac{1\,219\,336}{51\,051\sqrt{\pi}}$	$\frac{38\,580\,553\,125}{71\,374\,471\,168}\pi^2$
7	$\frac{745\,039}{24\,576}$	$\frac{2\,828\,819\,953\,125}{8796\,093\,022\,208}\pi^3$
8	$\frac{25\,796\,624\,240}{200\,783\,583\sqrt{\pi}}$	$\frac{10\,202\,766\,423\,046\,875}{15\,969\,609\,677\,012\,992}\pi^3$
9	$\frac{214\,422\,265}{1171\,456}$	$\frac{549\,540\,812\,759\,765\,625}{1288\,029\,493\,427\,961\,856}\pi^4$
10	$\frac{15\,033\,906\,553\,126}{17\,468\,171\,721\sqrt{\pi}}$	$\frac{3\,567\,616\,496\,493\,767\,578\,125}{3793\,868\,231\,748\,622\,483\,456}\pi^4$

$$\gamma_n \equiv \frac{\Gamma(3n+1/2)}{\Gamma(n+1/2)}\frac{1}{(36)^n n!}. \tag{B.4}$$

Similarly, for b_n

$$b_n = 4(\sqrt{2})^{-n}\frac{\Gamma(1/2)n!}{\Gamma\left(\frac{3n-1}{2}\right)}K_n \tag{B.5}$$

$$K_n \equiv \frac{3n-4}{4}K_{n-1} + \sum_{j=1}^{n-1} K_j K_{n-j}. \tag{B.6}$$

Putting these together and rearranging slightly, we find that the amplitudes C_n are given by

$$C_n = \frac{n!}{\Gamma\left(\frac{3n+1}{2}\right)}(R_n + 2K_n). \tag{B.7}$$

We tabulate the first ten values of C_n, along with the universal amplitude ratios $g_n = C_n / C_2^{n-1}$ in table B1.

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